Scaling for random walks on Eden trees

F. D. A. Aarão Reis*
Instituto de Física, Universidade Federal Fluminense, Avenida Litorânea s/n, Campus da Praia Vermelha, 24210-340 Niterói, Rio de Janeiro, Brazil

(Received 14 June 1996)

Random walks are simulated on finite stages of construction of Eden trees in dimensions $D=2$ and 3, and it is shown that the mean-square displacement $\langle R_N^2\rangle$ of N-step walks and the mean number of distinct visited sites $\langle S_N\rangle$ obey finite-size scaling. Accurate estimates of the dimensions of the random walks D_w are obtained and the relation $\langle S_N\rangle \sim N^{D/D_w}/(\log N)^\alpha$ is shown to hold in these fractals, with positive exponents α. Then the Alexander-Orbach scaling relation $D_s=2D/D_w$ is satisfied, where D_s is the spectral dimension, contrary to previous proposals in these and other treelike structures. [S1063-651X(96)50110-8]

PACS number(s): 05.40.+j, 05.50.+q

In 1982 Alexander and Orbach mapped the random walk problem onto the problem of vibrations with scalar elasticity on a fractal system [1], and obtained the scaling relation

$$D_s=2D_F/D_w, \tag{1}$$

where D_s is the spectral dimension of vibrations, D_w is the random walk dimension, and D_F is the fractal dimension of the substrate.

In the random walk problem, D_w is defined in the asymptotic behavior of the mean-square displacement of N-step walks

$$\langle R_N^2\rangle \sim N^{2/D_w}. \tag{2}$$

D_s appears in the asymptotic form of the mean number of distinct sites visited by the walker (when $D_s<2$) [2]:

$$\langle S_N\rangle \sim N^{D_s/2}. \tag{3}$$

Relation (1) is obtained considering that the fractal dimension of the region visited by the walker is equal to D_F, indicating an isotropic probability distribution of finding it within the traveled distance. It is a well accepted relation between dynamical and static exponents; see, for instance, the review articles [3] and [4]. It may be used, for example, to model real self-similar structures whose D_s is known from experiments [4].

Many efforts have been done to test relation (1) on several fractals [5–10]. In treelike structures, such as Eden trees and diffusion-limitted aggregates (DLAs), results of numerical simulations suggested that it was violated [7–9]. However, the results of Nakanishi and Herrmann [8] on Eden trees indicated a crossover from a short-time regime to a long-time different one, and the asymptotic behavior was not completely understood. Thus the validity of relation (1) in any fractal is controversial. The aim of this work is showing that it is valid in the Eden trees with logarithmic corrections in Eq. (3), then explaining the divergences found in these and other structures.

Recently it was proved that relation (1) is valid in a class of deterministic fractals, the Sierpinski carpets, if logarithmic corrections are incorporated in Eq. (3) [10]:

$$\langle S_N\rangle \sim \frac{N^{D_s/2}}{(\log N)^\alpha}, \tag{4}$$

where α is a positive exponent depending on D_F and the lacunarity. This asymptotic behavior resembles the two-dimensional case, where $\alpha=1$ and $D_s=2$ [11]. This result discarded a previous proposal of violation of relation (1) in the carpets [6], based on an analysis which did not consider the correction in Eq. (4).

In this work we study random walks on Eden trees in two and three dimensions. $\langle R_N^2\rangle$ and $\langle S_N\rangle$ are calculated for random walks confined on finite stages of construction of these lattices, and analyzed using finite-size scaling techniques, which separate finite-size effects and the true critical behavior in a convenient way [12]. The reliability of this technique was proved when applied to finitely ramified fractals where D_w is exactly known [13]. We will show that the asymptotic relations (2) and (4) are valid for those fractals, with D_w and D_s satisfying the Alexander-Orbach scaling relation (1). We will also present estimates of D_w and the logarithmic correction exponents α.

Eden trees are constructed by modifying the conventional Eden aggregation process [14]. Starting from one occupied site on a D-dimensional lattice, sequential growth occurs by additional occupation of one of the perimeter sites (randomly chosen) at each time step. In the conventional process, the perimeter sites are those which neighbor at least one occupied site. However, in the Eden trees, empty sites which neighbor more than one site become ineligible for occupation [7]. It gives rise to compact structures (dimension $D_F=D$) with no loops.

In $D=2$ we constructed 40 trees with volumes (number of sites) $V=10^4$, 2×10^4, and 3×10^4, and in $D=3$ we constructed 40 trees with $V=2\times10^4$, 3×10^4, and 4×10^4. On each of those finite aggregates, 10^5 random walks were simulated, with initial sites randomly chosen over the lattice. The maximum number of steps were 5×10^4 and 2×10^4 on the largest lattices in $D=2$ and $D=3$, respectively. Then $\langle R_N^2\rangle_V$ and $\langle S_N\rangle_V$ are estimated for N-step walks confined on

*Electronic address: reis@if.uff.br

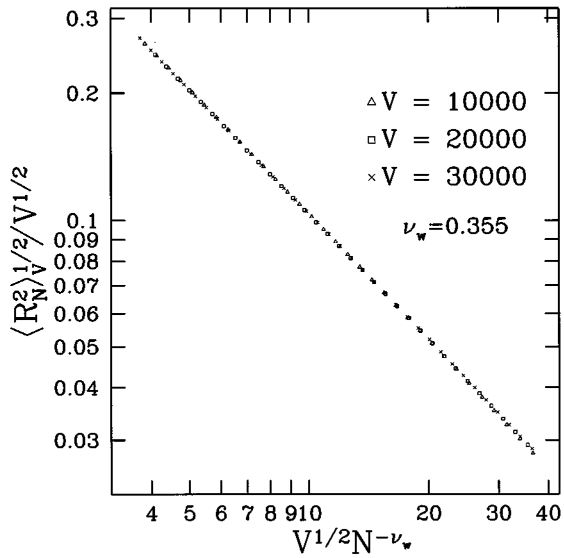

FIG. 1. Plot of $y = \langle R_N^2 \rangle_V^{1/2}/V^{1/2}$ versus $x = V^{1/2}N^{-\nu_w}$ for random walks on finite stages of construction of two-dimensional Eden trees.

those aggregates (the brackets indicate means over both the ensemble of walks and the ensemble of aggregates with volume V). The accuracy of these estimates is around 1%.

We propose the finite-size scaling relation

$$\langle R_N^2 \rangle_V^{1/2} \approx V^{1/D} f(V^{1/D} N^{-\nu_w}), \qquad (5)$$

where f is some function of the variable $x = V^{1/D}N^{-\nu_w}$, and

$$\nu_w = 1/D_w. \qquad (6)$$

This relation was already verified in deterministic fractals embedded in two and three-dimensional lattices (Sierpinski gaskets, carpets, and pastry shells) [13] .

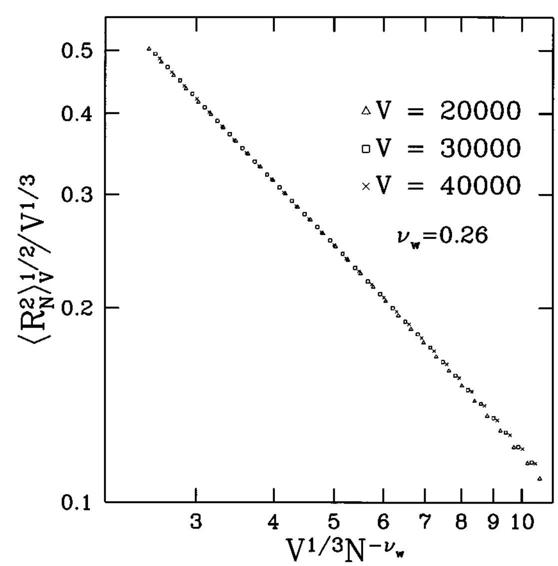

FIG. 2. Plot of $y = \langle R_N^2 \rangle_V^{1/2}/V^{1/3}$ versus $x = V^{1/3}N^{-\nu_w}$ for random walks on finite stages of construction of three-dimensional Eden trees.

TABLE I. Random walk dimensions (D_w), spectral dimensions (D_s), and logarithmic correction exponents α for $\langle S_N \rangle$ in Eden trees embedded in D-dimensional hypercubic lattices.

D	D_w	D_s	α
2	2.82 ± 0.06	1.42 ± 0.04	0.8 ± 0.1
3	3.85 ± 0.15	1.56 ± 0.06	0.9 ± 0.1

The plots of $y = \langle R_N^2 \rangle_V^{1/2}/V^{1/D}$ versus x are curves for each V which collapse into a single curve when the correct value of ν_w is chosen. In Figs. 1 and 2 we show the plots of some data for $D = 2$ and $D = 3$, respectively, with the best estimates of ν_w. The collapse of data for different V is obtained for walks with $N > 50$ in $D = 2$ and $N > 200$ in $D = 3$ ($x \approx 25$ and $x \approx 7$ for the smallest trees, respectively). It indicates that for small N we cannot observe the true asymptotic behaviour and, as we have to deal with data for large N in finite lattices, it is essential to use finite-size scaling to obtain reliable results.

In Table I we show the corresponding estimates of D_w. Together with relation (1) they give the estimates of D_s also shown in Table I. The previous best estimates of D_w were 2.47 ± 0.03 in $D = 2$ and 4.08 ± 0.09 in $D = 3$ [8], obtained using data of different small aggregates analyzed separately.

According to finite-size scaling, the same variable x is expected to describe the finite-size behavior of other physical quantities. For the mean number of distinct visited sites we propose the relation

$$\langle S_N \rangle_V \approx \frac{V}{(\log N)^\alpha} g(V^{1/D} N^{-\nu_w}). \qquad (7)$$

When $V \to \infty$ it is reasonable that $\langle S_N \rangle$ does not depend on V, thus $g(x) \sim x^{-D}$ when $x \to \infty$. Then Eq. (4) with D_s given by Eq. (1) is satisfied.

In Figs. 3 and 4 we plot $z = \langle S_N \rangle (\ln N)^\alpha / V$ versus x in two-

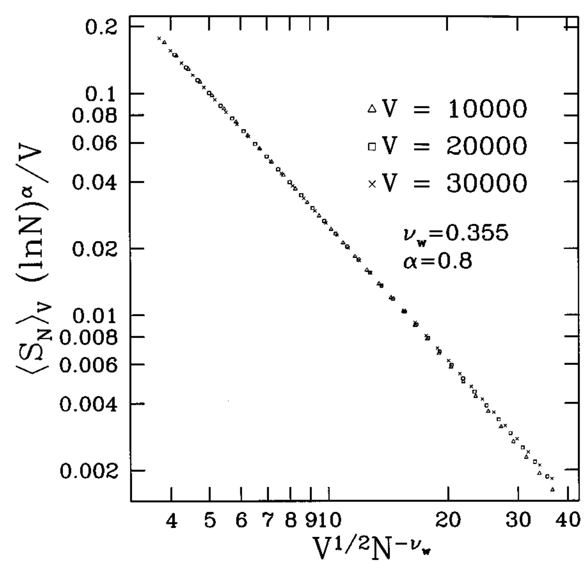

FIG. 3. Plot of $z = \langle S_N \rangle_V (\ln N)^\alpha / V$ versus $x = V^{1/2}N^{-\nu_w}$ for random walks on finite stages of construction of two-dimensional Eden trees.

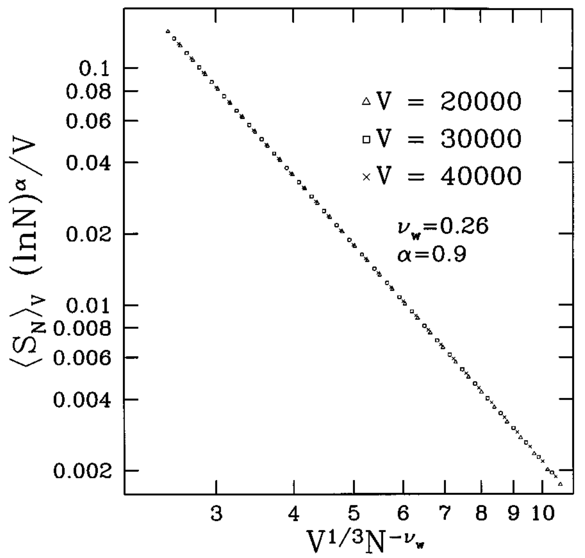

FIG. 4. Plot of $z=\langle S_N\rangle_V(\ln N)^\alpha/V$ versus $x=V^{1/3}N^{-\nu_w}$ for random walks on finite stages of construction of three-dimensional Eden trees.

and three-dimensional Eden trees, respectively. The exponents ν_w are the previous central estimates, and the logarithmic correction exponents α are the ones which provide the best collapses of curves for different V. Note that deviations from a single curve occur only for large x in the smallest lattices, corresponding to small N.

Those plots prove the validity of relation (7) for Eden trees and consequently the validity of the asymptotic form (4) with D_s given by the scaling relation (1). The final estimates of the exponents α are also shown in Table I.

An alternative finite-size scaling relation

$$\langle S_N\rangle_V \approx V^\gamma g(V^{1/D}N^{-\nu_w}) \tag{8}$$

would give the asymptotic form (3) for $\langle S_N\rangle$ with $D_s=2\gamma D/D_w$, which would not agree with relation (1) if $\gamma\neq 1$. This hypothesis is not supported by our data for any value of γ both in $D=2$ and $D=3$, confirming the asymptotic form (4).

From Eqs. (1), (2), and (4) we obtain a relation between the volume visited by the walker V_{vis} and the accessible volume V_{ac} within a radius R:

$$\frac{V_{vis}}{V_{ac}} \sim \frac{1}{(\log R)^\alpha}. \tag{9}$$

As previously observed [6–9], the density of the set of visited sites decreases faster than the density of accessible sites, but, as shown above, it occurs only with a logarithmic ratio between these densities. Then the original assumption ($V_{vis}\sim V_{ac}$) is not correct, but the essential ingredients of the problem are captured and the final result (Eq. (1)) is valid. It is also explained why an effective spectral dimension $D_s^{eff}<D_s$ must be found fitting data to Eq. (3). The comparison of the estimates of D_s in Table I and the previous ones [7,8] confirms this trend.

The logarithmic corrections in $\langle S_N\rangle$ were already expected in two-dimensional Eden trees since an exponent $\alpha=1$ is present in the exact solution of two-dimensional Euclidean lattices. The correction in three-dimensional Eden trees, however, is a surprising result. It opens the question of the dependence of exponent α on the fractal geometry. The results in some Sierpinski lattices (α in the range $0.1-0.5$) suggested that it was more important in infinitely ramified fractals [10], but the large values in the Eden trees, which are finitely ramified, discard this hypothesis. Further investigations will be necessary to find the origin of these corrections.

Previous estimates of D_s in DLAs also indicated a value smaller than $2D_F/D_w$ [9]. Similar investigations would be interesting in these structures, which model various experimental aggregates. Based on the discussion above, we expect that it will confirm once more the scaling relation (1).

[1] S. Alexander and R. Orbach, J. Phys. (Paris) Lett. **43**, L625 (1982).

[2] In Ref. [1] it is shown that D_s appears in the asymptotic form of the probability of the walker returning to its original site $P_0(N)\sim N^{-D_s/2}$. Considering that after a large number of steps there is an equal probability for the walker to be found in any visited site, Eq. (3) is obtained.

[3] S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987).

[4] T. Nakayama, K. Yakubo, and R. L. Orbach, Rev. Mod. Phys. **66**, 381 (1994).

[5] J. C. Angles d'Auriac, A. Benoit, and R. Rammal, J. Phys. A **16**, 4039 (1983).

[6] R. Dasgupta, T. K. Balabh, and S. Tarafdar, Phys. Lett. A **187**, 71 (1994).

[7] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **54**, 1346 (1985).

[8] H. Nakanishi and H. J. Herrmann, J. Phys. A **26**, 4513 (1993).

[9] D. J. Jacobs, S. Mukherjee, and H. Nakanishi, J. Phys. A **27**, 4341 (1994).

[10] F. D. A. Aarão Reis, Phys. Lett. A **214,** 239 (1996).

[11] E. W. Montroll and B. J. West, in *Fluctuation Phenomena*, edited by E. W. Montroll and J. L. Lebowitz (North-Holland, Amsterdam, 1979).

[12] M. N. Barber, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, New York, 1983), Vol. 8.

[13] F. D. A. Aarão Reis, J. Phys. A **28**, 6277 (1995).

[14] M. Eden, in *Proceedings of the Fourth Berkeley Symposium in Mathematics, Statistics and Probability*, edited by J. Neyman (University of California Press, Berkeley, 1961).